Int J Theor Phys (2008) 47: 1885–1890
DOI 10.1007/s10773-007-9633-6

Asymptotic Symmetries in Cosmological Models

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Received: 2 October 2007 / Accepted: 5 December 2007 / Published online: 8 January 2008

Abstract In the present article we study asymptotic isometries and asymptotic conformal Killing motions of some anisotropic Bianchi cosmological models. We show that asymptotically isotropic homogeneous cosmological spaces can be covariantly studied and characterized using scalars constructed with the help the Killing equations. We exhibit some illustrative examples of space-times possessing asymptotic Killing collineations. We apply our results in order to discuss asymptotic symmetries associated with scalar cosmological perturbations.

Keywords Asymptotic symmetries · Cosmology

1 Introduction

The search for of symmetries in cosmological models has been in the last years an active theme of research. After the publication of the general results of Katzin *et al.* [1, 2], about different types of symmetries of the space-time and Einstein field equations, many articles and monographes have been written on the problem of finding non-noetherian symmetries in general relativity [3–6]. Most of them deal with the classification of symmetries in different kinds of homogeneous Bianchi models [7–12].

The advantage of the Friedmann-Robertson-Walker over other cosmological models lies in its mathematical simplicity. The presence of six Killing vectors as well as of some collineations permits one to integrate the field equations and, in many cases, to impose auxiliary conditions to the source of gravity [5, 10, 13–16]. The presence of a conformal Killing motion in the Friedmann-Robertson Walker metric permits one to introduce the concept of

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temperature [17] and the relation between energy and temperature for a given equation of state. Although the notion of temperature and the $3+1$ slicing of the FRW metric in terms of conformal Killing motions cannot be straightforwardly extended to other cosmological metrics, it is possible to consider this symmetry as asymptotic provided that the cosmological model isotropizes to a FRW line-element.

The anisotropy of the CMB radiation as well as the matter distribution of the universe indicate that an isotropic and homogeneous cosmological metric is an approximation of a more realistic space-time describing the universe [18]. In the present article we show some examples of space-times that asymptotically evolve to a Friedmann Robertson Walker metric. We show that, in some cases it is possible to introduce covariant criteria of proximity in order to define temperature in asymptotically Friedmann Robertson Walker metrics.

Instead of classifying and studying the symmetry groups of approximate Friedmann-Robertson Walker metric, we proceed to construct geometric invariants with the help of the motions and collineations of the Friedmann-Robertson Walker metric. The scalars constructed using the almost FRW metric permit us to impose asymptotic conditions on the metric.

The article is structured as follows. In Sect. 2 we discuss some methods for computing asymptotic symmetries in almost Friedmann-Robertson-Walker metrics. In Sect. 3 we study the asymptotic symmetries of a spatially closed homogeneous cosmological universe with shear and null vorticity. In Sect. 4 we discuss the asymptotic symmetries of a perturbed spatially-flat Friedmann-Robertson-Walker universe.

2 Asymptotic Symmetries

Symmetries of the space-time can be characterized by the existence of Killing vectors ξ satisfying the equation,

$$L_{\xi} g_{\alpha\beta} = 0. \tag{1}$$

Spatially isotropic and homogeneous spaces in $3+1$ dimensions possess six Killing vectors, this maximal number of symmetries characterizes the FRW spaces. The existence of non-noetherian symmetries imposes further restrictions on the FRW metric, giving as a result the existence of conservation laws of the matter fields [3, 4]. The existence of a time-like conformal Killing motion satisfying the relation

$$\mathcal{L}_{\xi} g_{\alpha\beta} = 2\sigma g_{\alpha\beta} \tag{2}$$

permits one a $3+1$ slicing of the space-time, where the three-dimensional hypersurface is orthogonal to the conformal Killing vector. The Friemann-Robertson-Walker line-element possesses a conformal time-like Killing vector of the form

$$\xi^{\alpha} = (\xi^0, 0, 0, 0) \tag{3}$$

with $\xi^0 = u^0/T$, where T is a function depending on the time that can be identified as the temperature of the perfect fluid [17]. The existence of a conformal Killing vector of the form (3) permits us to write the space-time metric in the form

$$ds^2 = dt^2 - a^2(t)(g_{ab}(x^i)dx^a dx^b) \tag{4}$$

with $1 \leq a \leq 3,\ 1 \leq b \leq 3$. The expansion factor $a(\tau)$, expressed in the conformal time τ, is related to the temperature T via the expression

$$T = \frac{1}{ka(\tau)}. \tag{5}$$

The existence of a conformal Killing vector of the form given by (3) guarantees that the shear and vorticity vanish [19]

$$\omega_\alpha = 0 \to \omega = 0, \qquad \sigma_{\alpha\beta} = 0 \to \sigma = 0. \tag{6}$$

The condition (6) in a spatially isotropic space-time implies that the line element is FRW. It should be pointed out that almost FRW spaces do not possess in general conformal Killing vectors or spatial isotropy, therefore the search for symmetries should be done approximately [20, 21].

An asymptotically FRW space-time possesses an asymptotic or exact conformal Killing vector and asymptotic spatial Killing vectors as the time parameter goes to infinity. Our purpose is to construct invariants from the asymptotic metric with the help of conformal Killing motions of the FRW metric.

Using the six Killing vectors of a FRW space-time, we can impose the asymptotic condition on the almost FRW spaces with the help the invariants

$$\lim_{t\to\infty} g^{\alpha\beta} L_\xi g_{\alpha\beta} = \lambda \to 0 \tag{7}$$

where ξ^α is a Killing vector of the FRW metric.

We can also impose the alternative asymptotic condition

$$\lim_{t\to\infty} \xi^\alpha \xi^\beta L_\xi g_{\alpha\beta} = \tilde{\lambda} \to 0. \tag{8}$$

Using the conformal Killing vector $\xi^\alpha = a(t)u^\alpha$ of the FRW metric, we can introduce the asymptotic conditions based on the invariants

$$\lim_{t\to\infty} g^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} = \frac{8}{a(t)} \frac{da(t)}{dt} \tag{9}$$

and

$$\lim_{t\to\infty} \xi^\alpha \xi^\beta \mathcal{L}_\xi g_{\alpha\beta} = 2a(t) \frac{da(t)}{dt}. \tag{10}$$

Equations (9) and (10) impose necessary conditions on the metric of a space-time to be asymptotically FRW. In the next sections we apply (9) and (10) to some anisotropic cosmological models.

3 A Cosmological Model with Shear

In this section we are going to study the asymptotic symmetries of the homogeneous metric

$$\begin{aligned} ds^2 = {} & (1 - a^2(t)\sin^2(r)\sin^2\theta\psi(r,t)^2)dt^2 - 2a(t)\sin^2(r)\sin^2\theta\psi(r,t)dtd\varphi \\ & - a(t)^2(dr^2 + \sin^2(r)d\theta^2 + \sin^2(r)\sin^2\theta d\varphi^2). \end{aligned} \tag{11}$$

The line element (11) is associated with a spatially closed asymptotic Friedmann-Robertson-Walker metric where the function $\psi(r,t)$ measures the anisotropy of the model [22]. Choosing the velocity vector u_α as $u_\alpha = (-1,0,0,0)$, we obtain that metric (11) does not have vorticity and the shear σ depends on $\psi(r,t)$ as

$$\omega_{\alpha\beta} = 0, \ \sigma^2 = \frac{1}{2}\sin^2(r)\sin^2(\theta)\left(\frac{\partial\psi(r,t)}{\partial r}\right)^2. \tag{12}$$

The shear invariant σ (12) does not give enough information about the time dependence of $\psi(r,t)$. The invariant

$$g^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} = 8\frac{da(t)}{dt} \tag{13}$$

where $\xi^\alpha = (a(t),0,0,0)$, gives the same result obtained in the computation of the invariant associated with homothetic motions in a Friedmann-Robertson-Walker metric. The invariant

$$\xi^\alpha\xi^\beta L_\xi g_{\alpha\beta} = 2a^2(t)\left(\frac{da(t)}{dt} - \sin^2(r)\sin^2(\theta)\left(2a^2(t)\frac{da(t)}{dt}\psi^2 + a^3(t)\psi\frac{d\psi(t)}{dt}\right)\right) \tag{14}$$

shows that the line element (11) exhibit a FRW asymptotic behavior given by (10), provided that the function $\psi(r,t)$ has the following asymptotic behavior as

$$\psi \to \frac{f(r)}{a^2(t)} \tag{15}$$

where $f(r) \to f_0$ as $r \to \infty$.

4 Spatially Flat Perturbed Robertson-Walker Metric

In this section we proceed to compute the asymptotic symmetries of a spatially flat Robertson-Walker perturbed with a scalar field.

$$ds^2 = (1+2\Phi)a^2(\eta)d\eta^2 - (1-2\Phi)a^2(\eta)(dx^2+dy^2+dz^2), \tag{16}$$

$$\omega \neq 0, \qquad \sigma \neq 0. \tag{17}$$

The line element (16) can be regarded as a spatially-flat inflationary scenario by a scalar potential Φ which is responsible for density perturbations [23] The invariant equations

$$g^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} = -4\frac{\partial\Phi}{\partial x}\frac{1+4\Phi}{1-4\Phi^2} \tag{18}$$

and

$$\xi^\alpha\xi^\beta L_\xi g_{\alpha\beta} = 2a^2(\eta)\frac{\partial\Phi}{\partial x} \tag{19}$$

where $\xi^\alpha = (0,1,0,0)$ is a Killing vector of the spatially flat FRW metric, indicate that

$$\lim_{\eta\to\infty}\frac{\partial\Phi(x^i,\eta)}{\partial x^i} = 0. \tag{20}$$

Choosing the conformal Killing vector $\xi^\alpha = (1,0,0,0)$ we obtain

$$\xi^\alpha \xi^\beta L_\xi g_{\alpha\beta} = 2a\frac{da}{d\eta} + 4a\frac{da}{d\eta}\Phi + 2a^2\frac{d\Phi}{d\eta}, \tag{21}$$

$$g^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} = \frac{4(2\frac{da}{d\eta} - 8\frac{da}{d\eta}\Phi^2 - a\frac{d\Phi}{d\eta} - 4a\frac{d\Phi}{d\eta}\Phi)}{a(1-4\Phi^2)}. \tag{22}$$

From (21) and (22) we have that the condition

$$\Phi \to \frac{f(x^i)}{a^2(\eta)} \tag{23}$$

for large values of η is enough in order to guarantee asymptotic isotropization of the metric (16).

5 Concluding Remarks

The results present in the present article show that the construction of invariants based on Killing vectors and conformal Killing vectors is a powerful tool in the computation and study of asymptotic symmetries of asymptotically Friedmann-Robertson-Walker spacetimes. The existence of an asymptotic time-like conformal Killing motion can be used in order to compute the temperature of the background field. The derivation of asymptotic symmetries can be extended to include perturbations of the asymptotic Killing vectors. This case could permit us to construct approximative metrics exhibiting realistic temperature anisotropies.

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